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Gravitating macroscopic media in general relativity and macroscopic gravity

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Summary. — The problem of construction of a continuous (macroscopic) matter model for a given point-like (microscopic) matter distribution in general relativity is formulated. The existing approaches are briefly reviewed and a physical analogy with the similar problem in classical macroscopic electrodynamics is pointed out. The procedure by Szekeres in the linearized general relativity on Minkowski background to construct a tensor of gravitational quadruple polarization by applying Kaufman's method of molecular moments for derivation of the polarization tensor in macroscopic electrodynamics and to derive an averaged field operator by utilizing an analogy between the linearized Bianchi identities and Maxwell equations, is analyzed. It is shown that the procedure has some inconsistencies, in particular, (1) it has only provided the terms linear in perturbations for the averaged field operator which do not contribute into the dynamics of the averaged field, and (2) the analogy between electromagnetism and gravitation does break upon averaging. A macroscopic gravity approach in the perturbation theory up to the second order on a particular background space-time taken to be a smooth weak gravitational field is applied to write down a system of macroscopic field equations: Isaacson's equations with a source incorporating the quadruple gravitational polarization tensor, Isaacson's energy-momentum tensor of gravitational waves and energy-momentum tensor of gravitational molecules and corresponding equations of motion. A suitable set of material relations which relate all the tensors is proposed.

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1. – Introduction

In using the Einstein equations for a matter distribution in the form of a set of point-like mass constituents, there is a problem of adequate application, or validity, of the Einstein equations when such a matter distribution is substituted by a continuous matter distribution while the field operator in the left-hand side of the equations is kept unchanged. This problem as it stands in cosmology(1) is called *the averaging problem* [2, 3, 4, 5, 6]. Indeed, let us consider the Einstein equations in the mixed form(2)

$$g^{\alpha\epsilon}r_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}g^{\mu\nu}r_{\mu\nu} = -\kappa t^{\alpha(\text{discrete})}_{\beta} \tag{1}$$

with

$$t^{\alpha(\text{discrete})}_{\beta}(x) = \sum_i t_{(i)}{}^{\alpha}_{\beta}[x - z_{(i)}(\tau_{(i)})] \tag{2}$$

where $t_{(i)}{}^{\alpha}_{\beta}$ is a energy-momentum tensor for a point-like mass moving along its world line $z^{\mu} = z^{\mu}_{(i)}(\tau_{(i)})$ parameterized by $\tau_{(i)}$ and i counts for the matter particles in the distribution (2). Changing the discrete matter distribution to a continuous (hydrodynamical) one in the right-hand side of (1), which is the standard approach in cosmology [2, 3, 4, 5, 6] made phenomenologically on the basis of assumption about the uniformity and isotropicity of distribution of galaxies, or cluster of galaxies, throughout the whole Universe, means an implicit averaging denoted here by $\langle\cdot\rangle$

$$t^{\alpha(\text{discrete})}_{\beta}(x) \to T^{\alpha(\text{hydro})}_{\beta}(x) = \left\langle \sum_i t_{(i)}{}^{\alpha}_{\beta}[x - z_{(i)}(\tau_{(i)})] \right\rangle . \tag{3}$$

Given a covariant averaging procedure $\langle\cdot\rangle$ for tensors on space-time, the averaging out of (1) with taking into account (3) brings

$$\langle g^{\alpha\epsilon}r_{\epsilon\beta}\rangle - \frac{1}{2}\delta^{\alpha}_{\beta}\langle g^{\mu\nu}r_{\mu\nu}\rangle = -\kappa T^{\alpha(\text{hydro})}_{\beta} . \tag{4}$$

An important point regarding the averaged equations (4) is that in this form they are just algebraic relations between components of the smoothed hydrodynamical energy-momentum tensor and the average products of the metric tensor by the Ricci tensor $\langle g^{\alpha\epsilon}r_{\epsilon\beta}\rangle$ and cannot therefore be taken as field equations. By splitting the products out as $\langle g^{\alpha\epsilon}r_{\epsilon\beta}\rangle = \langle g^{\alpha\epsilon}\rangle\langle r_{\epsilon\beta}\rangle + C^{\alpha}_{\beta}$ where C^{α}_{β} is a correlation tensor, the averaged equations (4) become

$$\langle g^{\alpha\epsilon}\rangle\langle r_{\epsilon\beta}\rangle - \frac{1}{2}\delta^{\alpha}_{\beta}\langle g^{\mu\nu}\rangle\langle r_{\mu\nu}\rangle = -\kappa T^{\alpha(\text{hydro})}_{\beta} - C^{\alpha}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}C^{\epsilon}_{\epsilon}. \tag{5}$$

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(1) For discussion on the other physical settings on general relativity facing the same problem see [1].
(2) The mixed form is preferable here for the reason that it contains only products of metric by curvature. On contrary, the covariant or contravariant forms of the Einstein equations have triple products of metric by metric by curvature.

Here $\langle g^{\alpha\beta}\rangle$ and $\langle r_{\alpha\beta}\rangle$ denote the averaged inverse metric and the Ricci tensors which are supposed to describe the gravitational field due to the matter distribution $T^{\alpha(\mathrm{hydro})}_{\beta}$. A simple important observation [7, 8] now is that the averaged Einstein equations (5) are still not "real" field equations - just a definition of the correlation tensor C^{α}_{β} as a difference between (4) and (5). The origin of this fundamental fact is that the average of the non-linear operator of (1) on the metric tensor $g_{\rho\sigma}$ is not equal in general([3]) to an operator of *the same form* on the average metric $\langle g_{\rho\sigma}\rangle$:

$$(6)\qquad \left\langle \left(g^{\alpha\epsilon} r_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} g^{\mu\nu} r_{\mu\nu} \right) [g_{\rho\sigma}] \right\rangle \neq \left(\langle g^{\alpha\epsilon}\rangle\langle r_{\epsilon\beta}\rangle - \frac{1}{2}\delta^{\alpha}_{\beta}\langle g^{\mu\nu}\rangle\langle r_{\mu\nu}\rangle \right) [\langle g_{\rho\sigma}\rangle] \ .$$

In order to return them the status of the field equations one must define the object C^{α}_{β} and find its properties using information outside the Einstein equations.

To resolve the averaging problem, and to consider it in a broader context as the problem of macroscopic description of gravitation, the approach of macroscopic gravity has been proposed [5, 7, 8, 11, 12, 13, 14, 15] (see [7, 8, 13] for discussion of the problem and references therein, [1] for discussion of the physical status of general relativity as either a microscopic or macroscopic theory of gravity). A covariant space-time volume averaging procedure for tensor fields [5, 11, 14], has been defined and proved to exist on arbitrary Riemannian space-times with well-defined properties of the averages. Upon utilizing the averaging scheme, the macroscopic gravity approach has shown that (*i*) averaging out Cartan's structure equations brings about the structure equations for the averaged (macroscopic) non-Riemannian geometry and the definition and the properties of the correlation tensor C^{α}_{β}, (*ii*) the averaged Einstein's equations (5) become then the macroscopic field equations and they must be supplemented by a set of differential equations for the correlation tensor, (*iii*) it is always possible to extract the field operator of the form (1) for the Riemannian macroscopic metric tensor $G_{\mu\nu}$ and its Ricci tensor $M_{\mu\nu}$ with all other non-Riemannian correlation terms going to the right-hand side of (5) to give geometric correction to the averaged energy-momentum tensor $T^{\alpha(\mathrm{hydro})}_{\beta}$. It is been also shown [7, 8, 15] that only in case of neglecting all correlations of the gravitational field the averaged equations (5) becomes the macroscopic Einstein equations with a continuous matter distribution

$$(7)\qquad G^{\alpha\epsilon} M_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} G^{\mu\nu} M_{\mu\nu} = -\kappa T^{\alpha(\mathrm{hydro})}_{\beta} \ ,$$

which reveals the physical status of using the standard procedure in cosmology of claiming (5) to be the Einstein equations (7) after substitution of the matter model (3). The physical meaning, dynamical role and magnitude of the gravitational correlations must be elucidated in various physical settings. There is some evidence that they cannot be negligible for cosmological evolution (see, for example [16] for an estimation of the age of Universe in a second order perturbation approach).

([3]) It should noted that the inequality (6) has been observed in all possible averaging settings, for example, for a volume space-time averaging in [2, 5], in the framework of a kinetic approach in [9] and for a statistical ensemble averaging in [10]. Relations between different averaging procedures are discussed in [7, 8].

2. – Macroscopic media in general relativity

Derivation of the macroscopic (averaged) Maxwell field operator in macroscopic electrodynamics is easily accomplished due its linear field structure and the main problem consists in the construction of models of macroscopic electromagnetic media (for example, diamagnetics, magnetics, waveguides, etc.) [17, 18, 19, 20], which relates to the structure of the averaged current. In general relativity the problem of construction of macroscopic gravitating medium models is hardly elaborated due to the following reasons: (a) existing mathematical and physical difficulties in establishing the form of the averaged (macroscopic) operator in (5) for the field equations of macroscopic gravity recedes the interest in development of macroscopic gravitating media; (b) posing on its own more or less realistic problem with a discrete matter creates mathematical and physical problems due to nonlinearity and non-trivial geometry of gravitation, to mention, for example, the N-body problem, the problem of statistical description of gravity, etc. and (c) being relied on physically motivated phenomenological arguments (uniformity, isotropy, staticity, etc.) most applications of general relativity deal with *effective* continuous media if even a starting physical model is discrete in its nature like in cosmology (see Section **1**) or in description of extended bodies in general relativity (see [1] for discussion of the physical status of general relativity).

The kinetic approach in physics is known to provide a general scheme for introduction of characteristics of continuous media with a *known* distribution function of a discrete configuration. But the advantages of such generality are often greatly weakened in particular applications by difficulties of solving the Boltzmann equation to find a distribution function of interest. This applies to a great extent to general relativity where despite the formulation of the general relativistic Boltzmann equation [21, 22] the kinetic approach still remains useful for general definitions and considerations rather than being a working tool (see, for example, [9]) to derive a specific model of a macroscopic medium.

In case of the macroscopic electrodynamics together with the volume space-time averaging on Minkowski space-time the formalism of statistical distribution functions has been utilized (see [19] and references therein) and it is of importance for the mathematically well-posed derivation of the macroscopic theory and the general structure of averaged current starting from microscopic electrodynamics of point-like moving charges. Further application of the macroscopic theory requires usually mainly phenomenological considerations to establish material relations between macroscopic average fields and induction field necessary to make an overdetermined system of macroscopic equations determined. A correct derivation of material relations is known to require [23] averaging the microscopic equations with a given microscopic matter model during accomplishing an averaging of the microscopic field equations. Though it is the only self-consistent way, the elaboration of such kind of approach still remains a challenge even for simple physical settings.

On the other hand, volume (space, time, space-time) averaging procedures maintain their importance, direct physical meaning in application to macroscopic settings and their extreme clearness and descriptiveness. A volume averaging is also known to be unavoidable in all macroscopic settings (including statistical approaches) [20, 23, 24, 25] and space-time averages of physical fields are known to have the physical meaning as directly measurable quantities [26, 27] (for discussion see [7], [8] and references therein). That greatly motivates and supports interest in applying approaches with various averaging schemes in physics despite corresponding (mostly mathematical, not physical) difficulties in the rigorous formulation of averaging procedures.

The paper aims to approach the problem of construction of gravitating macroscopic media in general relativity by using an appropriate space-time averaging scheme.

3. – Szekeres' gravitational polarization tensor

The approach of Szekeres [28] to construct a tensor of gravitational quadruple polarization is based on the following assumptions: (a) the linearized theory of gravity on Minkowski background; (b) the linearized field equations are taken as the linearized Bianchi identity to employ an analogy between gravitation and electromagnetism; (c) the covariant method of molecular moments of Kaufman is applied to construct a tensor of quadruple gravitational polarization.

The equations under consideration are the contracted Bianchi identities

$$C_{\mu\nu\rho\sigma}{}^{;\sigma} = \kappa J_{\mu\nu\rho} \tag{8}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor interpreted as free gravitational field [29, 30], $J_{\mu\nu\rho}$ is a kind of "matter current" for the energy-momentum tensor $t_{\mu\nu}$

$$J_{\mu\nu\rho} = J_{[\mu\nu]\rho} = -(t_{\rho[\mu;\nu]} - \frac{1}{3} g_{\rho[\mu} t_{,\nu]}), \tag{9}$$

$$J_{\mu\nu\rho}{}^{;\rho} = 0. \tag{10}$$

equations (8) are analogous the Maxwell equations

$$f_{\mu\nu}{}^{;\nu} = \frac{4\pi}{c} j_\mu \tag{11}$$

with (10) being comparable with the conservation of the electromagnetic current j_μ

$$j_\mu{}^{;\mu} = 0. \tag{12}$$

Let us consider a number of particles labeled by i and having masses m_i and which are moving in there own effective gravitational field along world lines $z_i^\mu(\tau_i)$. A physical parameter which characterizes such a distribution is a typical characteristic distance l between neighbouring particles. Then the corresponding microscopic energy-momentum tensor has the form

$$t^{(\mathrm{micro})\mu\nu}(x) = c^{-1} \sum_i \int m_i \frac{dz_i^\mu}{d\tau_i} \frac{dz_i^\nu}{d\tau_i} \delta^4[x - z_i^\mu(\tau_i)] d\tau_i. \tag{13}$$

Assume now that due to gravitation the particles form into groups, a kind of gravitational molecules, which will be labeled by index a. From the physical point of view that means the presence of another parameter which is a characteristic size (diameter) L of such a molecule with $L \gg l$. It is a longwave *macroscopic* parameter and its presence in a microscopic system will be defining the dynamics of the system on the distances of order of L. The microscopic energy-momentum tensor (13) becomes now

$$t^{(\mathrm{molec})\mu\nu}(x) = c^{-1} \sum_a \sum_i^{ina} \int m_i \frac{d\tau_a}{d\tau_i} \frac{dz_i^\mu}{d\tau_a} \frac{dz_i^\nu}{d\tau_a} \delta^4[x - y_a^\mu(\tau_a) - s_i(\tau_a)] d\tau_a \tag{14}$$

where $y_a^\mu(\tau_a)$ is a world line of the a-th molecule center of mass [28] and $s_i^\mu(\tau_a) = z_i^\mu(\tau_i) - y_a^\mu(\tau_a)$ is a vector connecting the i-th particle with the center of mass of the molecule including this particle. Let us apply now the method of molecular moments of Kaufman [31] to represent (14) as a series expansion in powers of s_i^μ under assumption that the effective gravitational field which is created by moving gravitational molecules is a weak field and the perturbations of the gravitational field due to relative oscillations of gravitating particles in molecules are small compared with the mean effective field. After averaging out over a typical size of the gravitational molecule (for an averaging procedure see [5, 11]) one gets([4]) in accordance with the Szekeres procedure

$$\langle t^{(\text{molec})\mu\nu}\rangle = T^{(\text{free})\mu\nu} + D^{\mu\nu\rho}{}_{,\rho} + Q^{\mu\nu\rho\sigma}{}_{,\rho\sigma} \tag{15}$$

where $T^{(\text{free})\mu\nu}$ is the energy-momentum tensor of molecules, which has the form similar to (13) with substitution i by a, $D^{\mu\nu\rho}$ is the tensor of gravitational dipole polarization that can be incorporated into the quadrupole term, and $Q^{\mu\nu\rho\sigma}$ the tensor of gravitational quadrupole polarization

$$Q^{\mu\nu\rho\sigma} = c^{-1}\langle\sum_a \int q_a^{\mu\nu\rho\sigma}\delta^4(x-y_a)d\tau_a\rangle, \tag{16}$$

which has the symmetries of the Riemann tensor. The expression for the covariant gravitational quadrupole moment $q_a^{\mu\nu\rho\sigma}$ is defined as

$$q_a^{\mu\nu\rho\sigma} = g_a^{\mu\nu}u_a^\rho u_a^\sigma - g_a^{\rho\nu}u_a^\mu u_a^\sigma - g_a^{\mu\sigma}u_a^\rho u_a^\nu + g_a^{\rho\sigma}u_a^\mu u_a^\nu + u_a^\mu h_a^{\rho\nu\sigma} - u_a^\rho h_a^{\mu\nu\sigma} + u_a^\nu h_a^{\sigma\mu\rho} - u_a^\sigma h_a^{\nu\mu\rho} + k_a^{\mu\nu\rho\sigma}, \tag{17}$$

where

$$g_a^{\mu\nu} = \sum_i m_i \frac{d\tau_a}{d\tau_i} s_i^\mu s_i^\nu, \tag{18}$$

$$h_a^{\mu\nu\rho} = \frac{2}{3}\sum_i m_i \frac{d\tau_a}{d\tau_i} s_i^\mu \left(\frac{ds_i^\nu}{d\tau_a}s_i^\rho - \frac{ds_i^\rho}{d\tau_a}s_i^\nu\right), \tag{19}$$

$$k_a^{\mu\nu\rho\sigma} = \frac{2}{3}\sum_i m_i \frac{d\tau_a}{d\tau_i}\left(\frac{ds_i^\mu}{d\tau_a}s_i^\rho \frac{ds_i^\nu}{d\tau_a}s_i^\sigma - \frac{ds_i^\mu}{d\tau_i}s_i^\rho s_i^\nu \frac{ds_i^\sigma}{d\tau_a}\right). \tag{20}$$

Upon averaging (8) over the typical size of gravitational molecule the following equations were obtained:

$$\langle C_{\mu\nu\rho\sigma}\rangle^{,\sigma} = \kappa\langle J^{(\text{micro})}_{\mu\nu\rho}\rangle \tag{21}$$

where

$$\langle J^{(\text{micro})}_{\mu\nu\rho}\rangle = -\langle t^{(\text{micro})}_{\rho[\mu}\rangle_{,\nu]} + \frac{1}{3}\eta_{\rho[\mu}\langle t^{(\text{micro})}\rangle_{,\nu]}, \tag{22}$$

([4]) No explicit averaging procedure had been used in [28] and averaged relations and equations were being written rather on the basis of heuristic considerations than a rigorous analysis.

or

$$P_{\mu\nu\rho\sigma} = \frac{1}{2}(-Q_{\rho\sigma\epsilon[\mu}{}^{,\epsilon} - \frac{1}{3}\eta_{\rho[\mu}Q^{\gamma}{}_{\sigma\gamma\epsilon}{}^{,\epsilon})_{,\nu]} \tag{23}$$

and

$$\langle J^{(\mathrm{micro})}_{\mu\nu\rho}\rangle = J^{(\mathrm{free})}_{\mu\nu\rho} - P_{\mu\nu\rho\sigma}{}^{,\sigma}. \tag{24}$$

The expression (24) is analogous to the expression for the averaged electromagnetic current $\langle j^{(\mathrm{micro})\mu}\rangle$ for a bunch of charged particles moving along their world lines in the effective electromagnetic field in accordance with the microscopic equation (11) with $j^{(\mathrm{micro})\mu}$ when particles are grouped into molecules [31]

$$\langle j^{(\mathrm{micro})\mu}\rangle = j^{(\mathrm{free})\mu} - cP^{\mu\nu}{}_{,\nu} \tag{25}$$

where the polarization tensor $P^{\mu\nu}$ is defined as an average of the quadruple polarization moment of the molecules $p_a^{\mu\nu}$ (see [31, 28] for details)

$$P^{\mu\nu} = \langle\sum_a \int d\tau_a p_a^{\mu\nu}\delta^4(x-y_a)\rangle. \tag{26}$$

Then equations (21) can be rewritten as the macroscopic equations([5])

$$E_{\mu\nu\rho\sigma}{}^{,\sigma} = \kappa J^{(\mathrm{free})}_{\mu\nu\rho} \tag{27}$$

for the gravitational induction tensor $E_{\mu\nu\rho\sigma}$ defined as

$$E_{\mu\nu\rho\sigma} = \langle C_{\mu\nu\rho\sigma}\rangle + \kappa P_{\mu\nu\rho\sigma}. \tag{28}$$

The macroscopic equations (27) are analogous to the Maxwell macroscopic equations obtainable by means of averaging the microscopic equations (11) with $j^{(\mathrm{micro})\mu}$ with taking into account (25)

$$H^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c}J^{(\mathrm{free})\mu} \tag{29}$$

for the electromagnetic induction tensor $H_{\mu\nu}$ defined as

$$H^{\mu\nu} = \langle f^{\mu\nu}\rangle + 4\pi P_{\mu\nu}. \tag{30}$$

Unfortunately, at this point the analogy between the electromagnetism and gravitation which holds on the level of (8), (10) and (11), (12) breaks. Indeed, the formal similarity of (27), (28) and (29), (30) does not possess the structural analogy between averaged electromagnetism and gravitation: (A) the gravitational induction tensor $E_{\mu\nu\rho\sigma}$ does not have any more the symmetries of the Weyl tensor compared with $H_{\mu\nu}$ keeping the symmetries of $f_{\mu\nu}$; (B) it is constructed from the second derivatives of the polarization

([5]) Gravitational macroscopic equations similar to (21) are known to have been proposed first in [32].

tensor $Q_{\mu\nu\rho\sigma}$ compared with the linear algebraic structure of the electromagnetic induction tensor $H_{\mu\nu}$ in terms of the polarization tensor $P_{\mu\nu}$ - it is thus impossible to proceed with the formulation of phenomenological material relations between $E_{\mu\nu\rho\sigma}$ and $\langle C_{\mu\nu\rho\sigma}\rangle$ as possible in electromagnetism (relations between $H_{\mu\nu}$ and $\langle f_{\mu\nu}\rangle$, or amongst the fields **E**, **D**, **B**, **H** and **J**).

Even more important issue is that analysis of the macroscopic field equation (21)

$$\underset{\mathcal{O}(e)}{\langle C_{\mu\nu\rho\sigma}\rangle^{,\rho}} = \kappa\langle J^{(\mathrm{micro})}_{\mu\nu\rho}\rangle = \underset{\mathcal{O}(1)}{\kappa J^{(\mathrm{free})}_{\mu\nu\rho}} - \underset{\mathcal{O}(e^2)}{P_{\mu\nu\rho\sigma}{}^{,\sigma}} , \tag{31}$$

where e is a parameter measuring the value of deviation from the flat space, requires one to put into agreement the orders of magnitude of all quantities and reveals that the linearized Weyl tensor should be zero under averaging $\langle C_{\mu\nu\rho\sigma}\rangle = 0$. The Weyl tensor must be estimated in the perturbation theory up to the second order as it was done for the matter current $J^{(\mathrm{micro})}_{\mu\nu\rho}$ in the left-hand side of (8). So considering some physical features of the polarization tensor neither the macroscopic equations (21), nor any other field equations had in fact been employed [28].

On the basis of the expression

$$\langle t^{(\mathrm{micro})}_{\mu\nu}\rangle = T^{(\mathrm{free})}_{\mu\nu} + \frac{1}{2}Q_{\mu\rho\nu\sigma}{}^{,\rho\sigma} \tag{32}$$

the following material relations have been suggested

$$Q_{i0j0} = \langle G_{ij}\rangle N = \epsilon_g C_{i0j0} \tag{33}$$

where N is the average number of molecules per unit volume, G_{ij} is the quadrupole moment of a molecule

$$G_{ij} = \int \rho(x)\delta x_i \delta x_j d^3x, \tag{34}$$

$\rho = \rho(x)$ is the matter density in molecules, δx_i is a vector between neighbouring particles of a molecule. The quantity ϵ_g has been called the gravitational dielectric constant and in Newtonian approximation found to be

$$\epsilon_g = \frac{1}{4}\frac{mA^2c^2}{\omega_0^2}N \tag{35}$$

where A is the average linear dimension of a typical molecule, m is the average mass of the molecules, ω_0^2 is a typical frequency of harmonically oscillating particles in molecules.

4. – Macroscopic gravity equations

The gravitational field created by the particles is defined by Einstein's equation

$$g^{\alpha\epsilon}r_{\epsilon\beta} - \frac{1}{2}\delta^\alpha_\beta g^{\mu\nu}r_{\mu\nu} = -\kappa t^{\alpha(\mathrm{micro})}_\beta . \tag{36}$$

The Einstein equations (36) for the distribution (14) are of the form:

$$g^{\alpha\epsilon} r_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} g^{\mu\nu} r_{\mu\nu} = -\kappa t^{\alpha(\mathrm{molec})}_{\beta}. \tag{37}$$

Averaging the left-hand side of the Einstein equations (37) following the Isaacson's high-frequency approximation approach [33, 34], using the averaging procedure [5, 11] (one can also use the Isaacson's averaging procedure [33, 34], see also [12]) and with taking into account (16) brings the averaged Einstein equations in the form:

$$R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(0)} = -\kappa (T^{(\mathrm{free})}_{\mu\nu} + T^{(\mathrm{GW})}_{\mu\nu} + \frac{1}{2} Q_{\mu\rho\nu\sigma}{}^{;\rho\sigma}), \tag{38}$$

where $T^{(\mathrm{GW})}_{\mu\nu}$ is Isaacson's energy-momentum tensor of gravitational waves [33, 34]. All members in equation (38) can be shown to be of the same order of magnitude $\mathcal{O}(1/L^2)$. The macroscopic equations (38) give the equations of motion for molecules([6])

$$T^{(\mathrm{free})\mu\nu}{}_{;\nu} = 0, \tag{39}$$

conservation of the energy-momentum of gravitational waves

$$T^{(\mathrm{GW})\mu\nu}{}_{;\nu} = 0, \tag{40}$$

and an identity for the gravitational polarization

$$Q_{\mu\nu\rho\sigma}{}^{;\nu\sigma\mu} = 0. \tag{41}$$

The system of equations (38)-(41) is underdetermined - there are 20 unknown components of the tensor of gravitational polarization. It is possible to formulate two natural material relations. The first relation is between the traceless part of the quadrupole polarization tensor

$$\widetilde{Q}_{\mu\rho\nu\sigma} = Q_{\mu\rho\nu\sigma} - \frac{1}{4} g_{\mu\nu} P_{\rho\sigma}, \tag{42}$$

where $P_{\rho\sigma} = Q^{\mu}{}_{\rho\mu\sigma}$, and traceless energy-momentum tensor of gravitational waves

$$\frac{1}{2}\widetilde{Q}_{\mu\rho\nu\sigma}{}^{;\rho\sigma} = \lambda T^{(\mathrm{GW})}_{\mu\nu}, \tag{43}$$

where $\lambda = \lambda(x)$. Relation (43) can be shown to be always valid in the geometrical optics limit.

The second material relation connects the remaining part of the polarization tensor $Q_{\mu\rho\mu\sigma}$, its trace $P_{\rho\sigma}$, with a projection of the curvature tensor on the world line of an observer (electric part of the curvature tensor)

$$P_{\rho\sigma} = \epsilon R^{(0)}_{\mu\rho\nu\sigma} u^{\mu} u^{\nu}, \tag{44}$$

([6]) Under assumption that the background metric in the left-hand side of (38) represents a weak gravitational field on the flat background one can use the covariant derivatives with respect the metric in all relations instead of partial derivatives with respect to the flat metric.

where u^μ is the observer 4-velocity (4-velocity of the molecule centre of mass) and $\epsilon = \epsilon(x)$. The relation (44) can be shown to lead to the correct expression for the 3-tensor of the average quadrupole gravitational moment (34) so that

$$P_{\mu\nu} = (P_{00} = 0, P_{0i} = 0, P_{ij} = \langle G_{ij} \rangle N). \tag{45}$$

Then the material relation (33) can be recovered in the form

$$Q_{i0j0} = \langle G_{ij} \rangle N = \epsilon_g R_{i0j0} \tag{46}$$

that gives $\epsilon = \epsilon_g$ with the gravitational dielectric constant ϵ_g defined by (35).

Thus the system of equations (38)-(40), (43), (44) is fully determined and can be used to find the gravitational and polarization fields for the macroscopic gravitating systems.

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